

Fig. 3.4. Count the number of shortest zigzag paths.

that connect it with A . In Fig. 3.4 a few numbers so obtained are marked (but you should have obtained these numbers and a few more by yourself—check them at least). Observe these numbers—do you notice something?

If you have enough previous knowledge you may notice many things. Yet even if you have never before seen this array of numbers displayed by Fig. 3.4 you may notice an interesting relation: any number in Fig. 3.4 that is different from 1 is the sum of two other numbers in the array, of its northwest and northeast neighbors. For instance,

$$4 = 1 + 3, \quad 6 = 3 + 3$$

You may discover this law by observation as a naturalist discovers the laws of his science by observation. Yet, after having discovered it, you should ask yourself: Why is that so? What is the reason?

The reason is simple enough. Consider three corners in your network, the points X , Y , and Z , the relative position of which is shown by Fig. 3.4: X is the northwest neighbor and Y the northeast neighbor of Z . If we wish to reach Z coming from A along a shortest path in the network, we must pass either through X or through Y . Once we have reached X , we can proceed hence to Z in just one way, and the same is true for proceeding from Y to Z . Therefore, the *total number of shortest paths from A to Z* is a sum of two terms: it *equals the number of shortest paths from A to X added to the number of those from A to Y* . This explains fully our observation and proves the general law.

Having clarified this basic point, we can extend the array of numbers in Fig. 3.4 by simple additions till we obtain the larger array in Fig. 3.5, the south corner of which yields the desired answer: we can read the magic word in Fig. 3.2 in exactly 252 different ways.

3.6. The Pascal triangle

By now the reader has probably recognized the numbers and their

peculiar configuration which we have examined in the foregoing section. The numbers in Fig. 3.4 are *binomial coefficients* and their triangular arrangement is usually called the *Pascal triangle*. (Pascal himself called it the “arithmetical triangle.”) Further lines can be added to the triangle of Fig. 3.4 and, in fact, it can be extended indefinitely. The array in Fig. 3.5 is a square piece cut out of a larger triangle.

Some of the binomial coefficients and their triangular arrangement can be found in the writings of other authors before Pascal's *Traité du triangle arithmétique*. Still, the merits of Pascal in this matter are quite sufficient to justify the use of his name.

(1) We have to introduce a suitable *notation* for the numbers contained in the Pascal triangle; this is a step of major importance. For us each number attached to a point of this triangle has a geometric meaning: it indicates the number of different shortest zigzag paths from the apex of the triangle to that point. Each of these paths passes along the same number of blocks, let us say n blocks. Moreover, all these paths agree in the number of blocks described in the southwesterly direction and in the number of those in the southeasterly direction. Let l and r stand for these numbers, respectively (l to the left and r to the right—of course, downward in both cases). Obviously

$$n = l + r$$

If we give any two of the three numbers n , l , and r , the third is fully determined and so is the point to which they refer. (In fact, l and r are the rectangular coordinates of the point with respect to a system the origin of which is the apex of the Pascal triangle; one of the axes points southwest,

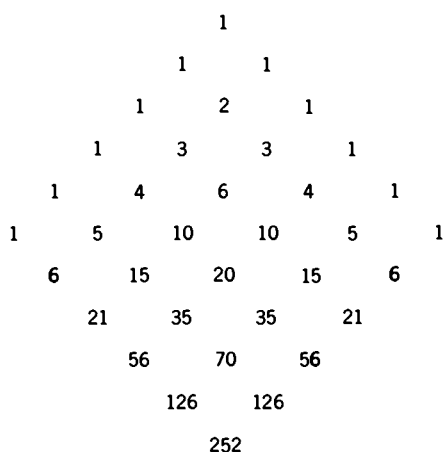


Fig. 3.5. A square from a triangle.

the other southeast.) For instance, for the last A of the path shown in Fig. 3.3

$$l = 5, \quad r = 5, \quad n = 10$$

and for the second B of the same path

$$l = 5, \quad r = 3, \quad n = 8$$

We shall denote by $\binom{n}{r}$ (this notation is due to Euler) the number of shortest zigzag paths from the apex of the Pascal triangle to the point specified by n (total number of blocks) and r (blocks to the right downward). For instance, see Fig. 3.5,

$$\binom{8}{3} = 56, \quad \binom{10}{5} = 252$$

The symbols for the numbers contained in Fig. 3.4 are assembled in Fig. 3.6. The symbols with the same number upstairs (the same n) are horizontally aligned (along the n th “base”—the base of a right triangle). The symbols with the same number downstairs (the same r) are obliquely aligned (along the r th “avenue”). The fifth avenue forms one of the sides of the square in Fig. 3.5—the opposite side is formed by the 0th avenue (but you may call it the borderline, or Riverside Drive, if you prefer to do so). The fourth base is emphasized in Fig. 3.4.

(2) Besides the geometric aspect, the Pascal triangle also has a computational aspect. All the numbers along the boundary (0th street, 0th avenue, and their common starting point) are equal to 1 (it is obvious that

$$\begin{array}{ccccccc}
 & & & & \binom{0}{0} & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
 \end{array}$$

$$\begin{array}{ccc}
 \binom{n}{r-1} & & \binom{n}{r} \\
 & \binom{n+1}{r} &
 \end{array}$$

Fig. 3.6. Symbolic Pascal triangle.

there is just one shortest path to these street corners from the starting point). Therefore,

$$\binom{n}{0} = \binom{n}{n} = 1$$

It is appropriate to call this relation the *boundary condition* of the Pascal triangle.

Any number inside the Pascal triangle is situated along a certain horizontal row, or base. We compute a number of the $(n + 1)$ th base by going back, or recurring, to two neighboring numbers of the n th base:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

see Fig. 3.6. It is appropriate to call this equation the *recursion formula* of the Pascal triangle.

From the computer's standpoint the numbers $\binom{n}{r}$ are determined (or defined, if you wish) by the recursion formula and the boundary condition of the Pascal triangle.

3.7. Mathematical induction

When we compute a number in the Pascal triangle by using the recursion formula, we have to rely on the previous knowledge of two numbers of the foregoing base. It would be desirable to have a scheme of computation independent of such previous knowledge. There is a well-known formula, which we shall call the *explicit formula* for binomial coefficients, that yields such an independent computation:

$$\binom{n}{r} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$$

Pascal's treatise contains the explicit formula (stated in words, not in our modern notation). Pascal does not say how he has discovered it and we shall not speculate too much how he might have discovered it. (Perhaps he just guessed it first—we often find such things by observation and tentative generalization of the observed; see the remark in the solution of ex. 3.39.) Yet Pascal gives a remarkable proof for the explicit formula and we wish to devote our full attention to his method of proof.⁴

We need a preliminary remark. The explicit formula does not apply,

⁴ Cf. Pascal's *Œuvres l.c.* footnote 3, pp. 455–464, especially pp. 456–457. The following presentation takes advantage of modern notation and modifies less essential details.

as it stands, to the case $r = 0$. Yet we lay down the rule that, if $r = 0$, it should be interpreted as

$$\binom{n}{0} = 1$$

The explicit formula does apply to the case $r = n$ and yields

$$\binom{n}{n} = \frac{n(n-1) \cdots 2 \cdot 1}{1 \cdot 2 \cdots (n-1)n} = 1$$

which is the correct result. Therefore, we have to prove the explicit formula only for $0 < r < n$, that is, in the interior of the Pascal triangle where we can use the recursion formula. Now, we quote Pascal, with unessential modifications some of which will be included in square brackets [].

Although this proposition [the explicit formula] contains infinitely many cases I shall give for it a very short proof, supposing two lemmas.

The first lemma asserts that the proposition holds for the first base, which is obvious. [The explicit formula is valid for $n = 1$, because, in this case, all possible values of r , $r = 0$ and $r = 1$, fall under the preliminary remark.]

The second lemma asserts this: if the proposition happens to be valid for any base [for any value n] it is necessarily valid for the next base [for $n + 1$].

We see hence that the proposition holds necessarily for all values of n . For it is valid for $n = 1$ by virtue of the first lemma; therefore, for $n = 2$ by virtue of the second lemma; therefore, for $n = 3$ by virtue of the same, and so on *ad infinitum*.

And so nothing remains but to prove the second lemma.

In accordance with the statement of the second lemma, we assume that the explicit formula is valid for the n th base, that is, for a certain value of n and all compatible values of r (for $r = 0, 1, 2, \dots, n$). In particular, along with

$$\binom{n}{r} = \frac{n(n-1) \cdots (n-r+2)(n-r+1)}{1 \cdot 2 \cdots (r-1) \cdot r}$$

we also have (if $r \geq 1$)

$$\binom{n}{r-1} = \frac{n(n-1) \cdots (n-r+2)}{1 \cdot 2 \cdots (r-1)}$$

Adding these two equations and using the recursion formula, we derive as a necessary consequence

$$\begin{aligned} \binom{n+1}{r} &= \binom{n}{r} + \binom{n}{r-1} = \frac{n(n-1) \cdots (n-r+2)}{1 \cdot 2 \cdots (r-1)} \left[\frac{n-r+1}{r} + 1 \right] \\ &= \frac{n(n-1) \cdots (n-r+2)}{1 \cdot 2 \cdots (r-1)} \cdot \frac{n+1}{r} \\ &= \frac{(n+1)n(n-1) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdots r} \end{aligned}$$

That is, the validity of the explicit formula for a certain value of n involves its validity for $n + 1$. This is precisely what the second lemma asserts—we have proved it.

The words of Pascal which we have quoted are of historic importance because his proof is the first example of a fundamental pattern of reasoning which is usually called *mathematical induction*.

This pattern of reasoning deserves further study.⁵ If carelessly introduced, reasoning by mathematical induction may puzzle the beginner; in fact, it may appear as a devilish trick.

You know, of course, that the devil is dangerous: if you give him the little finger, he takes the whole hand. Yet Pascal's second lemma does exactly this: by admitting the first lemma you give just one finger, the case $n = 1$. Yet then the second lemma also takes your second finger (the case $n = 2$), then the third finger ($n = 3$), then the fourth, and so on, and finally takes all your fingers even if you happen to have infinitely many.

3.8. Discoveries ahead

After the work in the three foregoing sections, we now have three different approaches to the numbers in the Pascal triangle, the binomial coefficients.

(1) *Geometrical approach.* A binomial coefficient is the number of the different shortest zigzag paths between two given corners in a network of streets.

(2) *Computational approach.* The binomial coefficients can be defined by their recursion formula and their boundary condition.

(3) *Explicit formula.* We have proved it, by Pascal's method, in sect. 3.7.

The name of the numbers considered reminds us of another approach.

(4) *Binomial theorem.* For indeterminate (or variable) x and any non-negative integer n we have the identity

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

For a proof, see ex. 3.1.

There are still other approaches to the numbers in the Pascal triangle which play, in fact, a role in a great many interesting questions and possess a great many interesting properties. "This table of numbers has eminent

⁵ HSI, Induction and mathematical induction, pp. 114–121; MPR, vol. 1, pp. 108–120.

3.32. Consider the sum of the numbers along a base of the Pascal triangle:

$$\begin{array}{rcl} 1 & & = 1 \\ 1 + 1 & & = 2 \\ 1 + 2 + 1 & & = 4 \\ 1 + 3 + 3 + 1 & & = 8 \end{array}$$

These facts seem to suggest a general theorem. Can you guess it? Having guessed it, can you prove it? Having proved it, can you devise another proof?

3.33. Observe

$$\begin{array}{rcl} 1 - 1 & & = 0 \\ 1 - 2 + 1 & & = 0 \\ 1 - 3 + 3 - 1 & & = 0 \\ 1 - 4 + 6 - 4 + 1 & & = 0 \end{array}$$

generalize, prove, and prove again.

3.34. Consider the sum of the first six numbers along the third avenue of the Pascal triangle:

$$1 + 4 + 10 + 20 + 35 + 56 = 126$$

Locate this sum in the Pascal triangle, try to observe analogous facts, generalize, prove, and prove again.

3.35. Add the thirty-six numbers displayed in Fig. 3.5, try to locate their sum in the Pascal triangle, formulate a general theorem, and prove it. (Adding so many numbers is a boring task—in doing it cleverly, you may easily catch the essential idea.)

3.36. Try to recognize and locate in the Pascal triangle the numbers involved in the following relation:

$$1 \cdot 1 + 5 \cdot 4 + 10 \cdot 6 + 10 \cdot 4 + 5 \cdot 1 = 126$$

Observe (or remember) analogous cases, generalize, prove, prove again.

3.37. Try to recognize and locate in the Pascal triangle the numbers involved in the following relation:

$$6 \cdot 1 + 5 \cdot 3 + 4 \cdot 6 + 3 \cdot 10 + 2 \cdot 15 + 1 \cdot 21 = 126$$

Observe (or remember) analogous cases, generalize, prove, prove again.

3.38. Fig. 3.8 shows the first four from an infinite sequence of figures each of which is an assemblage of equal circles into an equilateral triangular shape. Any circle that is not on the rim of the assemblage touches six surrounding circles. In the n th figure there are n circles aligned along each side of the triangular assemblage and the total number of circles in this n th figure is termed the n th *triangular number*. Express the n th triangular number in terms of n and locate it in the Pascal triangle.

3.39. Replace in Fig. 3.8 each circle by a sphere (a marble) of which the

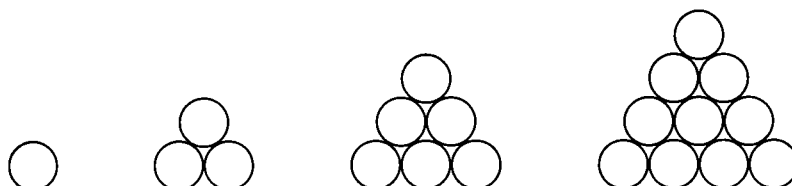


Fig. 3.8. The first four triangular numbers.

circle forms the equator. Fix 10 marbles arranged as in Fig. 3.8 on a horizontal plane, place 6 marbles on top (they fit neatly into the interstices) as a second layer, add 3 marbles on top of these as a third layer and place finally 1 marble on the very top. This configuration of

$$1 + 3 + 6 + 10 = 20$$

marbles is so related to a regular tetrahedron as each of the assemblages of circles shown by Fig. 3.8 is related to a certain equilateral triangle: 20 is the fourth *pyramidal number*. Express the n th pyramidal number in terms of n and locate it in the Pascal triangle.

3.40. You can build a pyramidal pile of marbles in another manner: begin with a layer of n^2 marbles, arranged in a square as in Fig. 3.9, place on top of it a second layer of $(n - 1)^2$ marbles, then $(n - 2)^2$ marbles, and so on, and finally just one marble on the very top. How many marbles does the pile contain?

3.41. Interpret the product

$$\binom{n_1}{r_1} \binom{n_2}{r_2} \binom{n_3}{r_3} \cdots \binom{n_h}{r_h}$$

as the number of a certain set of zigzag paths in a network of streets.

3.42. All the shortest zigzag paths from the apex of the Pascal triangle to the point specified by n (the total number of blocks) and r (blocks to the right downward) have a point in common with the line of symmetry of the Pascal triangle (from the first A to the last A in Fig. 3.3) namely their common initial

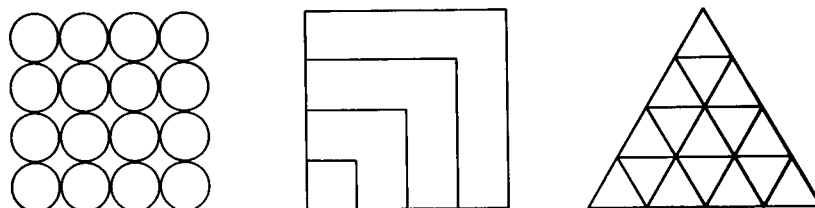


Fig. 3.9. The fourth square number.

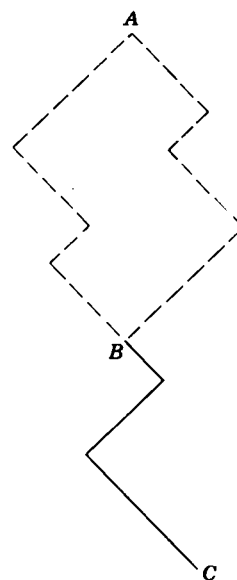


Fig. 3.11. A modification of the decisive idea.

derive hence that

$$N = \frac{|2r - n|}{n} \binom{n}{r}$$

3.43. (Continued) The number of all shortest zigzag paths from the apex to the n th base, that have only the initial point in common with the line of symmetry, is $\binom{2m}{m}$ if $n = 2m$ is even and $2 \binom{2m}{m}$ if $n = 2m + 1$ is odd.

3.44. Trinomial coefficients. Fig. 3.12 shows a fragment of an infinite triangular array of numbers defined by two conditions.

(1) *Boundary condition.* Each horizontal line or "base" (this term has been similarly used in sect. 3.6) begins with 0, 1 and ends with 1, 0. (The n th base consists of $2n + 3$ numbers and so the boundary condition leaves undefined $2n - 1$ numbers of the n th base, for $n = 1, 2, 3, \dots$)

				0	1	0						
				0	1	1	1	0				
			0	1	2	3	2	1	0			
		0	1	3	6	7	6	3	1	0		
	0	1	4	10	16	19	16	10	4	1	0	
0	1	5	15	30	45	51	45	30	15	5	1	0

Fig. 3.12. Trinomial coefficients.

(2) *Recursion formula.* Any number of the $(n + 1)$ th base left undefined by (1) is computed as the sum of three numbers of the n th base: of its northwestern, northern, and northeastern neighbors. (For instance, $45 = 10 + 16 + 19$.)

Compute the numbers of the seventh base. (They are, with three exceptions, divisible by 7.)

3.45. (Continued) Show that the numbers of the n th base, beginning and ending with 1, are the coefficients in the expansion of $(1 + x + x^2)^n$ in powers of x . (Hence the name “trinomial coefficient.”)

3.46. (Continued) Explain the symmetry of Fig. 3.12 with respect to its middle vertical line.

3.47. (Continued) Observe that

$$\begin{array}{rcl} 1 + 1 + 1 & & = 3 \\ 1 + 2 + 3 + 2 + 1 & & = 9 \\ 1 + 3 + 6 + 7 + 6 + 3 + 1 & & = 27 \end{array}$$

generalize and prove.

3.48. (Continued) Observe that

$$\begin{array}{rcl} 1 - 1 + 1 & & = 1 \\ 1 - 2 + 3 - 2 + 1 & & = 1 \\ 1 - 3 + 6 - 7 + 6 - 3 + 1 & & = 1 \end{array}$$

generalize and prove.

3.49. (Continued) Observe that the value of the sum

$$1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 19$$

is a trinomial coefficient, generalize, and prove.

3.50. (Continued) Find lines in Fig. 3.12 agreeing with lines in the Pascal triangle.

3.51. *Leibnitz's Harmonic Triangle.* Fig. 3.13 shows a section of this little known but remarkable arrangement of numbers. It has properties which are so to say “analogous by contrast” to those of the Pascal triangle. That triangle contains integers, this one (as far as visible) the reciprocals of integers. In Pascal's triangle, each number is the sum of its northwestern and northeastern neighbors. In Leibnitz's triangle, each number is the sum of its southwestern and southeastern neighbors; for instance

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}, \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{12}, \quad \frac{1}{6} = \frac{1}{12} + \frac{1}{12}$$

This is the *recursion formula* of the Leibnitz triangle. This triangle has also a *boundary condition*: the numbers along the northwest borderline (the “0th avenue”) are the reciprocals of the successive integers, $1/1, 1/2, 1/3, \dots$. (The boundary condition of the Pascal triangle is of a different nature: values are prescribed along the whole boundary, 0th avenue and 0th street.)